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ORIGINAL PAPER

# The effect of nested grid sampling on the parameter estimation of a spatial Gompertz diffusion

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**Abstract** This paper evaluates the effects of using data observed on regular nested grids on the parameter estimates of a two-parameter Gompertz diffusion model. This new spatial diffusion process represents a technically more complex stage of Gompertz modeling. Firstly, the diffusion model is introduced through an appropriate transformation of a two-parameter Gaussian diffusion process. Probabilistic characteristics of this model, such as the transition densities and the trend functions, are obtained. Secondly, statistical estimation is considered using data obtained on a regular or irregular grid; the explicit expression of the likelihood equations and the parameter estimators are given for regular grids. Finally, a simulation experiment illustrates the results of this paper.

**Keywords** Diffusion process · Gompertz diffusion process · Maximum likelihood · Spatial process

## 1 Introduction

The Gompertz stochastic model was introduced by Prajenshu (1980) and Tan (1986), and applied by several authors (see, for example, Troynikov 1998; Miller et al. 2000). The diffusion version of this constitutes a model of great interest to investigators in several fields, including demography, biology, economics and environmental sciences. This process is known in the literature as the stochastic Gompertz growth model and is applied in diffusion schemes for modelling positive (or negative) feedback processes (see Gutiérrez et al. 2006a, b). In economics terms, they are applied for modelling processes in which cheap products become cheaper and expensive products become more expensive, a phenomenon known as *winner takes all* (see McGee and Sammut 2002) or *winner takes most* (see Amit and Zott 2001). In such cases a diffusion model with a sigmoidal development (for example, a Gompertz diffusion) is considered and then appropriate competitive interaction terms are included.

In the one-parameter case, important real phenomena have been successfully modeled using Gompertz diffusions. In that respect, Kiiski and Pohjola (2002) applied them for analysing the Internet diffusion between 1995 and 2000. In the theory of population growth, Ricciardi (1977) applied a Gompertz diffusion by adding a white noise fluctuation to the intrinsic fertility of a population. In energy studies, Gutiérrez et al. (2005a, b) used them for explaining the growth of the natural gas consumption in Spain in comparison with other models. Recently, in cell growth studies, Albano and Giorno (2006) considered a Gompertz model for modeling the tumor growth and, in environmental sciences, Gutiérrez et al. (2008a) used a bivariate stochastic Gompertz diffusion for modeling the gross domestic product and $CO_2$ emissions in Spain.

From the point of view of the statistical inference on Gomperz diffusions, the problem of estimating the parameters in the drift coefficient has great interest, among other reasons, because a diffusion process can be introduced by means of a stochastic differential equation (Arnold 1973). The drift coefficient of this equation might include “exogenous factors” that let us consider known external variables which could have a hypothetical influence over the endogenous variable. This approach gives alternatives in relation

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to parameter estimation and inference, derivation of prediction and simulation schemes and other analyses.

Random field models have been considered to solve problems, among other areas of application, in hydrogeology, geostatistics, climate modeling, or environmental analysis (see, for example, Christakos 1992). Two-parameter Gompertz diffusions are suitable to model growth phenomena on a subset of the plane. Different authors introduce the Gompertz diffusion process from the point of view of Itô's stochastic differential equations and the stochastic model is then solved analytically by applying Itô's calculus. In fact, the two-parameter Gompertz diffusion considered in this paper could be introduced as the solution of a stochastic partial differential equation (SPDE) by applying Proposition 2.4 established in Nualart (1983), which would require to prove the hypotheses I to IV stated in that paper. However, taking into account that other characteristics such as the transition densities and problems as parameter estimation are solved by considering the Gompertz diffusion as certain transformation of a Gaussian diffusion, in the next section, the spatial Gompertz diffusion process will be introduced through an appropriate transformation of a two-parameter Gaussian diffusion.

Technically, the Lognormal diffusion model is a particular case of a Gompertz diffusion. Gutiérrez and Roldán (2007) carried out a general study of two-parameter Lognormal diffusions that Gutiérrez et al. (2005a, b, 2007) completed with the development of techniques for estimation, prediction and conditional simulation of these diffusions. Using these results, in the second section several analytical properties for the two-parameter Gompertz diffusion targeted at making inference in this model using discrete sampling are obtained. The problem of estimating the parameters involved in the model is dealt with in the third section considering the maximum likelihood methodology and using data observed on a regular or irregular grid. In the Gompertz model, data are affected by an exponential function and then the very high variability between them implies that with few data the estimates obtained will be very poor. Therefore, in the fourth section a simulation experiment is developed to illustrate the effect of regular nested grid sampling schemes on the parameter estimates.

## 2 Two-parameter Gompertz diffusion model

The theory of two-parameter diffusions was introduced by Nualart (1983) considering a class of spatial processes which are diffusions on each coordinate and satisfy a particular Markov property related to partial ordering in $\mathbf{R}_+^2$.

Let $\{X(\mathbf{z}) : \mathbf{z} = (s,t) \in I = [0,S] \times [0,T] \subset \mathbf{R}_+^2\}$ be a positive valued two-parameter Markov process (see Cairoli 1971) defined on a probability space $(\Omega, \mathcal{A}, P)$, with initial value $X(0,0) = 1$. The distribution of the spatial process is determined by the following transition probabilities:

$$\begin{aligned} &P(B,(s+h,t+k) \mid (x_1,x,x_2),\mathbf{z}) \\ &\quad = P[X(s+h,t+k) \in B \mid X(s,t+k) = x_1, \\ &(\mathbf{z}) = x, X(s+h,k) = x_2], \end{aligned}$$

where $\mathbf{z} = (s,\ t) \in I,\ h,\ k > 0,\ \mathbf{x} = (x_1,\ x,\ x_2)$ and $B$ is a Borel subset. We suppose that the transition densities exist and are given by

$$\begin{aligned} &g(y,(s+h,t+k) \mid \mathbf{x},\mathbf{z}) \\ &\quad = \frac{1}{y\sqrt{2\pi v^2_{\mathbf{z};h,k}}} \exp\left\{ -\frac{1}{2}\left( \frac{\ln(y) - \mu_{\mathbf{z},\mathbf{x};h,k}}{v_{\mathbf{z};h,k}} \right)^2 \right\}, \end{aligned}$$

for $y \in \mathbf{R}_+$, with

$$\begin{aligned} \mu_{\mathbf{z},\mathbf{x};h,k} &= e^{-\beta_1 h}\ln x_1 + e^{-\beta_2 k}\ln x_2 - e^{-\beta_1 h - \beta_2 k}\ln x \\ &\quad + \left(\alpha - \frac{\sigma^2}{2}\right)\frac{1}{\beta_1\beta_2}\left(1 - e^{-\beta_1 h} - e^{-\beta_2 k} + e^{-\beta_1 h - \beta_2 k}\right), \\ v^2_{\mathbf{z};h,k} &= \frac{\sigma^2}{4\beta_1\beta_2}\left(1 - e^{-2\beta_1 h} - e^{-2\beta_2 k} + e^{-2\beta_1 h - 2\beta_2 k}\right). \end{aligned}$$

and $\alpha$, $\beta_1$, $\beta_2$ and $\sigma$ being real parameters.

Let $\{Y(\mathbf{z}) : \mathbf{z} \in I\}$ be the spatial process defined as

$$Y(\mathbf{z}) = e^{\boldsymbol{\beta}\mathbf{z}^T}\ln X(\mathbf{z}).$$

The initial value is $Y(0,\ 0) = 0$ and the transition densities are given by

$$\begin{aligned} &f(y,(s+h,t+k) \mid (x_1,x,x_2),\mathbf{z}) \\ &= \frac{1}{y\sqrt{2\pi v^2_{\mathbf{z};h,k}}} \exp\left\{ -\frac{1}{2}\left( \frac{y - x_1 - x_2 + x - m_{\mathbf{z};h,k}}{v_{\mathbf{z};h,k}} \right)^2 \right\}, \end{aligned}$$

for $x_1,\ x_2,\ x,\ y \in \mathbf{R},\ \mathbf{z} = (s,\ t) \in I,\ h,\ k > 0$, with

$$\begin{aligned} m_{\mathbf{z};h,k} &= \left(\alpha - \frac{\sigma^2}{2}\right)\int_s^{s+h}\int_t^{t+k} e^{\beta_1\sigma + \beta_2\tau}\, d\sigma\, d\tau \\ &= \left(\alpha - \frac{\sigma^2}{2}\right)\frac{1}{\beta_1\beta_2}\left(e^{\beta_1(s+h)} - e^{\beta_1 s}\right)\left(e^{\beta_2(t+k)} - e^{\beta_2 t}\right), \\ v^2_{\mathbf{z};h,k} &= \sigma^2\int_s^{s+h}\int_t^{t+k} e^{2\beta_1\sigma + 2\beta_2\tau}\, d\sigma\, d\tau \\ &= \sigma^2\frac{1}{4\beta_1\beta_2}\left(e^{2\beta_1(s+h)} - e^{2\beta_1 s}\right)\left(e^{2\beta_2(t+k)} - e^{2\beta_2 t}\right). \end{aligned}$$

Under these conditions (see Gutiérrez and Roldán, 2007), $\{Y(\mathbf{z}) : \mathbf{z} \in I\}$ is a two-parameter Gaussian diffusion with drift and diffusion coefficients, respectively, given by

$$\tilde{a}(s,t) = \left(\alpha - \frac{\sigma^2}{2}\right)e^{\boldsymbol{\beta}\mathbf{z}^T}, \quad \tilde{B}(s,t) = \sigma^2 e^{2\boldsymbol{\beta}\mathbf{z}^T}.$$

The remaining diffusion coefficients are all null. Furthermore, if $\mathbf{z}, \mathbf{z}' \in I$, $\mathbf{z} = (s, t)$, $\mathbf{z}' = (s', t')$, then

$$m_Y(\mathbf{z}) = E[Y(\mathbf{z})] = \int_0^s \int_0^t \tilde{a}(\sigma,\tau)d\sigma d\tau = \left(\alpha - \frac{\sigma^2}{2}\right)\frac{1}{\beta_1\beta_2}\left(e^{\beta_1 s} - 1\right)\left(e^{\beta_2 t} - 1\right),$$

$$\sigma_Y^2(\mathbf{z}) = var(Y(\mathbf{z})) = \int_0^s \int_0^t \tilde{B}(\sigma,\tau)d\sigma d\tau = \frac{\sigma^2}{4\beta_1\beta_2}\left(e^{2\beta_1 s} - 1\right)\left(e^{2\beta_2 t} - 1\right),$$

$$c_Y(\mathbf{z},\mathbf{z}') = cov(Y(\mathbf{z}),Y(\mathbf{z}')) = \sigma_Y^2(\mathbf{z}\wedge\mathbf{z}'),$$

where we write $\mathbf{z} \wedge \mathbf{z}'$ for $(s \wedge s', t \wedge t')$, with '$\wedge$' denoting the minimum. Therefore, we can finally assert that $\{X(\mathbf{z}) : \mathbf{z} \in I\}$ is a two-parameter Gompertz diffusion.

Taking into account that $Y(\mathbf{z}) \rightsquigarrow N\left(m_Y(\mathbf{z}), \sigma_Y^2(\mathbf{z})\right)$, it is clear that

$$e^{-\boldsymbol{\beta}\mathbf{z}^t}Y(\mathbf{z}) \rightsquigarrow N\left(e^{-\boldsymbol{\beta}\mathbf{z}^t}m_Y(\mathbf{z}), e^{-2\boldsymbol{\beta}\mathbf{z}^t}\sigma_Y^2(\mathbf{z})\right)$$

and then, the trend functions of the two-parameter Gompertz diffusion are given by the following expression:

$$\begin{aligned}E[X(\mathbf{z})] &= E\left[\exp\left(e^{-\boldsymbol{\beta}\mathbf{z}^t}Y(\mathbf{z})\right)\right]\\ &= \exp\left\{e^{-\boldsymbol{\beta}\mathbf{z}^t}m_Y(\mathbf{z}) + \frac{e^{-2\boldsymbol{\beta}\mathbf{z}^t}}{2}\sigma_Y^2(\mathbf{z})\right\}\\ &= \exp\left\{\left(\alpha - \frac{\sigma^2}{2}\right)\frac{1}{\beta_1\beta_2}\left(1 - e^{-\beta_1 s}\right)\left(1 - e^{-\beta_2 t}\right)\right.\\ &\quad\left. + \frac{\sigma^2}{8\beta_1\beta_2}\left(1 - e^{-2\beta_1 s}\right)\left(1 - e^{-2\beta_2 t}\right)\right\}.\end{aligned}$$

The class of two-parameter diffusions under consideration satisfy that the stochastic processes which appear fixing each coordinate of the parameter space are diffusions as well, in this case, Gompertz diffusions. Next, we summarize some interesting results related to these processes.

For fixed $t \in [0, T]$, the stochastic process $\{Y(s, t) : s \in [0, S]\}$ is a Gaussian diffusion with drift and diffusion coefficients given by

$$\tilde{a}_{1,t}(s) = \int_0^t \tilde{a}(s,\tau)d\tau = \left(\alpha - \frac{\sigma^2}{2}\right)e^{\beta_1 s}\int_0^t e^{\beta_2\tau}d\tau = \left(\alpha - \frac{\sigma^2}{2}\right)\frac{e^{\beta_1 s}}{\beta_2}\left(e^{\beta_2 t} - 1\right),$$

$$\tilde{B}_{1,t}(s) = \int_0^t \tilde{B}(s,\tau)d\tau = \sigma^2 e^{2\beta_1 s}\int_0^t e^{2\beta_2\tau}d\tau = \frac{\sigma^2 e^{2\beta_1 s}}{2\beta_2}\left(e^{2\beta_2 t} - 1\right).$$

By means of the transformation $X(s,t) = \exp\left(e^{-\boldsymbol{\beta}\mathbf{z}^t}Y(s,t)\right)$, and using Itô's stochastic calculus, the stochastic process $\{X(s, t) : s \in [0, S]\}$ is the Gompertz diffusion with drift and diffusion coefficients given, respectively, by

$$a_{1,t}(s,x) = \left(-\beta_1 \ln x + \left(\alpha - \frac{\sigma^2}{2}\right)\frac{e^{\beta_1 s}}{\beta_2}\left(e^{\beta_2 t} - 1\right) + \frac{\sigma^2 e^{2\beta_1 s}}{2\beta_2}\left(e^{2\beta_2 t} - 1\right)\right)x$$

$$B_{1,t}(s,x) = \frac{\sigma^2 e^{2\beta_1 s}}{2\beta_2}\left(e^{2\beta_2 t} - 1\right)x^2.$$

For fixed $s \in [0, S]$, the stochastic process $\{X(s, t) : t \in [0, T]\}$ satisfies similar properties which can be obtained by symmetry. Finally, note that for $\boldsymbol{\beta} = \mathbf{0} = (0,0)$, $X(\mathbf{z})$ is a two-parameter Lognormal diffusion (see Gutiérrez and Roldán 2007).

## 3 Statistical inference on the model

In this section, the maximum likelihood method is applied to obtain the estimates of $\alpha$, $\beta_1$, $\beta_2$ and $\sigma^2$; firstly, when data are assumed to be observed on a regular grid, and secondly, when data are observed on an irregular grid. A numerical example is described for illustrating the two-parameter Gompertz diffusion model and the solution of the likelihood equations using data on a regular grid.

### 3.1 Parameter estimation from data on a regular grid

Let us suppose that data are observed on a regular grid with $n = m_1 m_2$ locations, and $c_1$ and $c_2$ are the constant increments on the $X$-axis and on the $Y$-axis, respectively, that is, data are observed on the spatial locations

$$\mathbf{z}_{ij} = (c_1 i, c_2 j), \quad i = 1,\ldots,m_1, \quad j = 1,\ldots,m_2.$$

Denoting

$$\mathbf{x} = \left\{x_{ij},\ i = 1,\ldots,m_1,\ j = 1,\ldots,m_2\right\},$$

$$\gamma = \alpha - \frac{\sigma^2}{2}, \theta_1 = e^{-c_1\beta_1} \quad \text{and} \quad \theta_2 = e^{-c_2\beta_2},$$

and taking into account that $\{P[x_{ij} = 1] = 1 : i = 0$ or $j = 0\}$, the joint density function is given by

$$\begin{aligned}L\left(\mathbf{x};\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\sigma}^2\right) &= \prod_{i=1}^{m_1}\prod_{j=1}^{m_2}\frac{1}{x_{ij}\sqrt{2\pi\frac{\sigma^2}{4\beta_1\beta_2}\left(1-\theta_1^2\right)\left(1-\theta_2^2\right)}}\\ &\quad\times\exp\left\{\frac{-2\beta_1\beta_2}{\sigma^2\left(1-\theta_1^2\right)\left(1-\theta_2^2\right)}\right.\\ &\quad\times\left(\ln x_{ij} - \theta_1\ln x_{i-1j} - \theta_2\ln x_{ij-1}\right.\\ &\quad\left.\left.+\theta_1\theta_2\ln x_{i-1j-1} - \frac{\gamma(1-\theta_1)(1-\theta_2)}{\beta_1\beta_2}\right)^2\right\}.\end{aligned}$$

Considering the log-likelihood function, differentiating it with respect to $\gamma$ and $\sigma^2$, and equating to 0, we obtain the estimates for $\gamma$ and $\sigma^2$:

$$\widehat{\gamma} = \frac{\hat{\beta}_1\hat{\beta}_2}{n\left(1-\hat{\theta}_1\right)\left(1-\hat{\theta}_2\right)} \times \left(\sum_{i,j}\ln x_{ij} - \hat{\theta}_1\sum_{i,j}\ln x_{i-1j} - \hat{\theta}_2\sum_{i,j}\ln x_{ij-1} + \hat{\theta}_1\hat{\theta}_2\sum_{i,j}\ln x_{i-1j-1}\right), \tag{1}$$

$$\hat{\sigma}^2 = \frac{4\hat{\beta}_1\hat{\beta}_2}{n\left(1-\hat{\theta}_1^2\right)\left(1-\hat{\theta}_2^2\right)}\sum_{i,j}\left[\ln x_{ij} - \hat{\theta}_1\ln x_{i-1j} - \hat{\theta}_2\ln x_{ij-1} + \hat{\theta}_1\hat{\theta}_2\ln x_{i-1j-1} - \widehat{\gamma}\left(1-\hat{\theta}_1\right)\left(1-\hat{\theta}_2\right)/\hat{\beta}_1\hat{\beta}_2\right]^2. \tag{2}$$

Differentiating the log-likelihood function with respect to $\beta_1$ and $\beta_2$, equating to 0, and using Eqs. 1 and 2, we obtain the following non-linear equation system:

$$\sum_{i,j}\left[\ln x_{ij} - \hat{\theta}_1\ln x_{i-1j} - \hat{\theta}_2\ln x_{ij-1} + \hat{\theta}_1\hat{\theta}_2\ln x_{i-1j-1} - \frac{\widehat{\gamma}\left(1-\hat{\theta}_1\right)\left(1-\hat{\theta}_2\right)}{\hat{\beta}_1\hat{\beta}_2}\right] \times \left(-\ln x_{i-1j} + \hat{\theta}_2\ln x_{i-1j-1}\right) = 0, \tag{3}$$

$$\sum_{i,j}\left[\ln x_{ij} - \hat{\theta}_1\ln x_{i-1j} - \hat{\theta}_2\ln x_{ij-1} + \hat{\theta}_1\hat{\theta}_2\ln x_{i-1j-1} - \frac{\hat{\gamma}\left(1-\hat{\theta}_1\right)\left(1-\hat{\theta}_2\right)}{\hat{\beta}_1\hat{\beta}_2}\right] \times \left(-\ln x_{ij-1} + \hat{\theta}_1\ln x_{i-1j-1}\right) = 0. \tag{4}$$

Replacing Eq. 1 in Eqs. 3 and 4, we obtain two equations that only depend on $\theta_1$ and $\theta_2$. This non-linear system is solved to obtain $\hat{\theta}_1$ and $\hat{\theta}_2$. As we can see in the examples, these equations have several solutions but only one of them is real (the remaining solutions are complex and then invalid).

### 3.1.1 Numerical example

For illustrating the computational solution of the likelihood Eqs. 1–4 a two-parameter Gompertz diffusion with $\beta_1 = 1$, $\beta_2 = 2$, $\gamma = 0.5$ (equivalently, $\alpha = 2.5$) and $\sigma^2 = 4$ was considered.

A MatLab program was implemented to carry out the calculations. Data were obtained by unconditional simulation on a square grid (see Fig. 1) with the unit measurement as the grid spacing and with SW corner at point (1,1) and NE corner at point (20, 20); see Fig. 2. Figure 3 shows the histograms of the 400 values of $\{X(\mathbf{z}_i)\}$ and $\{\ln X(\mathbf{z}_i)\}$.

The values of the maximum likelihood estimates of $\beta_1$ and $\beta_2$ were obtained as follows. Equations 3 and 4 are:

$$-26876.8 + 76194.0\hat{\theta}_1 + 6142.4\hat{\theta}_2 - 25156.0\hat{\theta}_2^2 - 19062.7\hat{\theta}_1\hat{\theta}_2 + 72381.4\hat{\theta}_1\hat{\theta}_2^2 = 0$$

$$-10032.1 + 6142.4\hat{\theta}_1 + 76030.5\hat{\theta}_2 - 9531.3\hat{\theta}_1^2 - 50312.0\hat{\theta}_1\hat{\theta}_2 + 72381.4\hat{\theta}_1^2\hat{\theta}_2 = 0$$

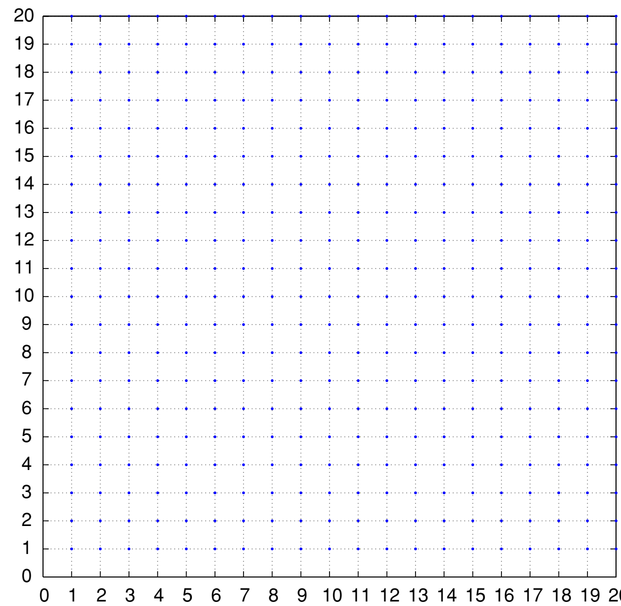


**Fig. 1** Regular grid with 20 × 20 observation locations

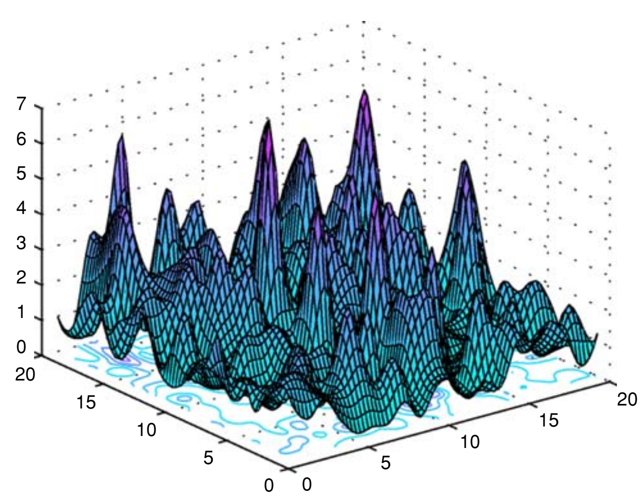


**Fig. 2** Contour-level plot of 400 values of $\{X(\mathbf{z}_i)\}$

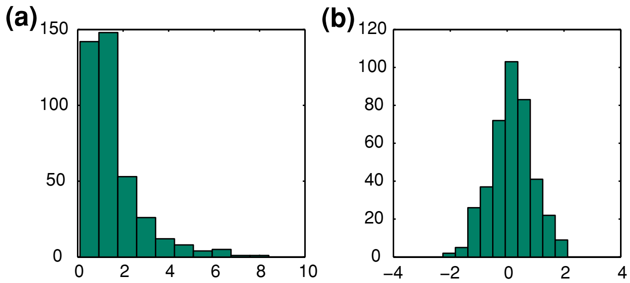


**Fig. 3** Histograms of 400 values of **a** $\{X(\mathbf{z}_i)\}$ and **b** $\{\ln X(\mathbf{z}_i)\}$

The real solution is:

$$\left\{\hat{\theta}_1 = 0.3537, \hat{\theta}_2 = 0.1345\right\},$$

and then

$$\left\{\hat{\beta}_1 = 1.0393, \hat{\beta}_2 = 2.0060\right\}$$

Replacing them in Eqs. 1 and 2, $\widehat{\gamma} = 0.3677$ and $\hat{\sigma}^2 = 4.1240$.

## 3.2 Parameter estimation from data on an irregular grid

Let us now suppose that data $\mathbf{X} = (X(\mathbf{z}_1),\ldots,X(\mathbf{z}_n))^t$ are observed at known spacial locations $\mathbf{z}_1 = (s_1,t_1),\ldots, \mathbf{z}_n = (s_n, t_n) \in I$, and let $\mathbf{x} = (x_1,\ldots,x_n)$ be a sample. Let us consider the transformed $n$-dimensional random vector

$$\mathbf{Y} = (Y(\mathbf{z}_1),\ldots,Y(\mathbf{z}_n))^t = \left(e^{\boldsymbol{\beta}\mathbf{z}_1^t}\ln X(\mathbf{z}_1),\ldots,e^{\boldsymbol{\beta}\mathbf{z}_n^t}\ln X(\mathbf{z}_n)\right)^t$$

and the transformed sample, $\mathbf{y} = (y_1,\ldots,y_n)$. Under the previous conditions we can assert that the joint density function of the random vector $\mathbf{Y}$ is

$$f(\mathbf{y}) = \frac{1}{(2\pi)^{n/2}|\Sigma_Y|^{1/2}}\exp\left\{-\frac{1}{2}(\mathbf{y}-\mathbf{m}_Y)^t(\Sigma_Y)^{-1}(\mathbf{y}-\mathbf{m}_Y)\right\},$$

where

$$\mathbf{m}_Y = (m_Y(\mathbf{z}_1),\ldots,m_Y(\mathbf{z}_n))^t,$$
$$\Sigma_Y = \left(\sigma_Y^2(\mathbf{z}_i \wedge \mathbf{z}_j)\right)_{i,j=1,\ldots,n}.$$

Denoting

$$\theta_{1i} = e^{\beta_1 s_i},\ i=1,\ldots,n,$$
$$\theta_{2i} = e^{\beta_2 t_i},\ i=1,\ldots,n,$$

$$\mathbf{E}_{\boldsymbol{\theta}} = \begin{pmatrix} e^{\boldsymbol{\beta}\mathbf{z}_1^t} & 0 & \cdots & 0 \\ 0 & e^{\boldsymbol{\beta}\mathbf{z}_2^t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\boldsymbol{\beta}\mathbf{z}_n^t} \end{pmatrix} = \begin{pmatrix} \theta_{11}\theta_{21} & 0 & \cdots & 0 \\ 0 & \theta_{12}\theta_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \theta_{1n}\theta_{2n} \end{pmatrix},$$

the mean and the covariance function of $\mathbf{Y}$ are, respectively, given by

$$m_Y(\mathbf{z}_i) = \left(\alpha - \frac{\sigma^2}{2}\right)\frac{1}{\beta_1\beta_2}\left(e^{\boldsymbol{\beta}\mathbf{z}^t} - e^{\beta_1 s} - e^{\beta_2 t} + 1\right)$$
$$= \frac{\gamma}{\beta_1\beta_2}(1-\theta_{1i})(1-\theta_{2i}),$$
$$\sigma_Y^2(\mathbf{z}_i) = \frac{\sigma^2}{4\beta_1\beta_2}\left(e^{2\boldsymbol{\beta}\mathbf{z}^t} - e^{2\beta_1 s} - e^{2\beta_2 t} + 1\right)$$
$$= \frac{\sigma^2}{4\beta_1\beta_2}\left(1-\theta_{1i}^2\right)\left(1-\theta_{2i}^2\right).$$

Let us write

$$\mathbf{m}_Y = \frac{\gamma}{\beta_1\beta_2}\mathbf{m}_{\boldsymbol{\theta}} = \frac{\gamma}{\beta_1\beta_2}\begin{pmatrix}(1-\theta_{11})(1-\theta_{21}) \\ (1-\theta_{12})(1-\theta_{22}) \\ \vdots \\ (1-\theta_{1n})(1-\theta_{2n})\end{pmatrix}$$

$$\Sigma_Y = \frac{\sigma^2}{4\beta_1\beta_2}\mathbf{C}_{\boldsymbol{\theta}}$$
$$= \sigma^2\left(\frac{1}{4\beta_1\beta_2}\left(1-\theta_{1(s_i\wedge s_j)}\theta_{1(t_i\wedge t_j)}\right)\right.$$
$$\left.\times\left(1-\theta_{2(s_i\wedge s_j)}\theta_{2(t_i\wedge t_j)}\right)\right)_{i,j=1,\ldots,n}.$$

With this notation, the joint density function of the random vector $\mathbf{X}$ is given by

$$g(\mathbf{x}) = \frac{\prod_{i=1}^n x_i}{(2\pi)^{n/2}\left|\frac{\sigma^2}{4\beta_1\beta_2}\mathbf{C}_{\boldsymbol{\theta}}\right|^{1/2}\prod_{i=1}^n \theta_{1i}\theta_{2i}}$$
$$\times\exp\left\{-\frac{2\beta_1\beta_2}{\sigma^2}\left(\mathbf{E}_{\boldsymbol{\theta}}\ln\mathbf{x} - \frac{\gamma}{\beta_1\beta_2}\mathbf{m}_{\boldsymbol{\theta}}\right)^t\mathbf{C}_{\boldsymbol{\theta}}^{-1}\right.$$
$$\left.\times\left(\mathbf{E}_{\boldsymbol{\theta}}\ln\mathbf{x} - \frac{\gamma}{\beta_1\beta_2}\mathbf{m}_{\boldsymbol{\theta}}\right)\right\}.$$

As before, considering the log-likelihood function, differentiating with respect to $\gamma$ and $\sigma^2$, and equating to 0, we obtain

$$\widehat{\gamma} = \hat{\beta}_1\hat{\beta}_2\left(\mathbf{m}_{\hat{\theta}}^t\mathbf{C}_{\hat{\theta}}^{-1}\mathbf{m}_{\hat{\theta}}\right)^{-1}\mathbf{m}_{\hat{\theta}}^t\mathbf{C}_{\hat{\theta}}^{-1}\mathbf{E}_{\hat{\theta}}\ln\mathbf{x}$$

and

$$\hat{\sigma}^2 = \frac{4\hat{\beta}_1\hat{\beta}_2}{n}\left(\mathbf{E}_{\hat{\theta}}\ln\mathbf{x} - \frac{\gamma}{\hat{\beta}_1\hat{\beta}_2}\mathbf{m}_{\hat{\theta}}\right)^t\mathbf{C}_{\hat{\theta}}^{-1}\left(\mathbf{E}_{\hat{\theta}}\ln\mathbf{x} - \frac{\hat{\gamma}}{\hat{\beta}_1\hat{\beta}_2}\mathbf{m}_{\hat{\theta}}\right).$$

The case of data observed on an irregular grid is technically more complex and explicit expressions of the non-linear equation system satisfied by $\beta_1$ and $\beta_2$ cannot be obtained. In this case, the implementation of the maximum likelihood methodology can be done directly by numerical computation.

## 4 Simulation experiment: Sensitivity of the parameter estimates to the density of nested grid data

A simulation experiment was carried out in order to evaluate the effect of the grid density on the estimates of the parameters $\beta_1$, $\beta_2$, $\gamma$ and $\sigma^2$. In this experiment a spatial Gompertz diffusion on a subset of the plane with $\beta_1 = 0.5$, $\beta_2 = 1$, $\gamma = 0.5$ and $\sigma^2 = 4$ was considered and data were simulated on nested square ($m_1 = m_2 = m$) grids: $X(\mathbf{z}_i) \equiv X(i, j), i, j = 1,\ldots,m$, with $m = 10, 20, 40, 80$ (see Fig. 4). The number of simulations was chosen to be $M = 10000$, which justifies the inclusion up to the third decimal place in Table 1. Figure 5 shows a boxplot for each parameter calculated using the estimates obtained from the 10000 replications of the simulation experiment. The average of these estimates and the estimated mean squared error (MSE) for each of them calculated as $(1/10000) \sum_{r=1}^{10000} (\theta - \hat{\theta}_r)^2$, are presented in Table 1. The computation of data by unconditional simulation, as well as the estimation of the parameters and MSEs were carried out using a MatLab program. Figure 5 shows how the efficiency of the estimates increases as the grid size increases.

## 5 Conclusion

The two-parameter Gompertz diffusion model introduced in this paper represents a technically more complex stage of Gompertz modeling. In a real case, data observed on a subset of the plane that exhibit non-normality and a very high variability can be successfully modeled, among other alternatives, as coming from a Gompertz spatial process.

Data are affected by an exponential function and then the very high variability between them implies that with few data the estimates obtained will be very poor. The worst results are obtained for $\sigma^2$, since its calculation depends on the results of the remaining estimates.

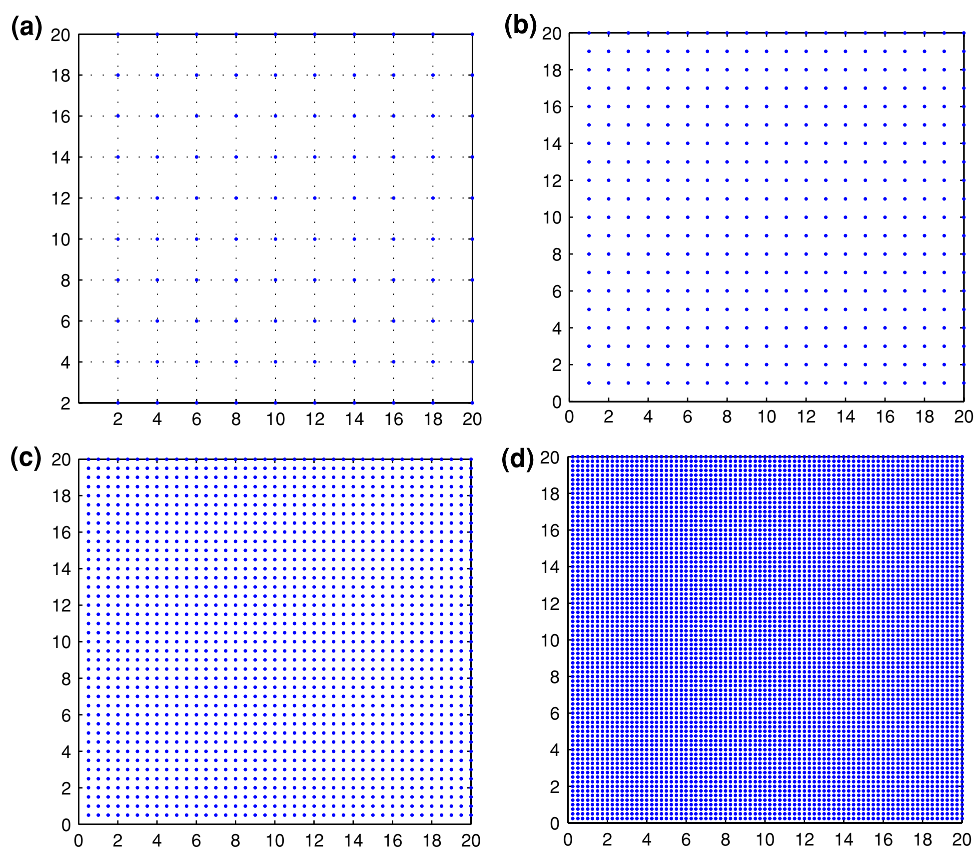


**Fig. 4** Spatial configuration of the simulation experiment. Dots show the data location: **a** sparse (10 × 10), **b** medium (20 × 20), **c** dense (40 × 40) and **d** very dense (80 × 80) nested grids

**Table 1** Monte Carlo estimates and estimated MSEs of the parameters

| $m$ | $\hat{\beta}_1$ | $MSE(\beta_1)$ | $\hat{\beta}_2$ | $MSE(\beta_2)$ | $\hat{\gamma}$ | $MSE(\gamma)$ | $\hat{\sigma}^2$ | $MSE(\sigma^2)$ |
|---|---|---|---|---|---|---|---|---|
| 10 | 0.549 | 0.027 | 1.135 | 0.187 | 0.636 | 1.132 | 3.599 | 1.525 |
| 20 | 0.513 | 0.005 | 1.023 | 0.019 | 0.522 | 0.022 | 3.824 | 0.261 |
| 40 | 0.506 | 0.002 | 1.006 | 0.005 | 0.507 | 0.013 | 3.951 | 0.038 |
| 80 | 0.503 | 0.001 | 1.003 | 0.002 | 0.505 | 0.011 | 3.987 | 0.007 |

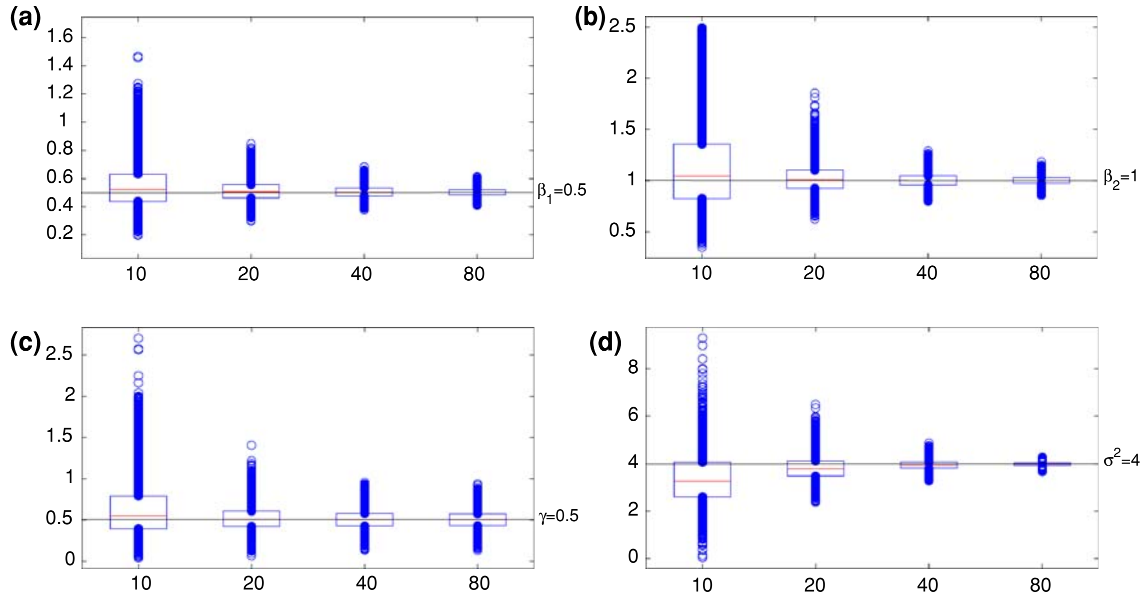


**Fig. 5** Boxplots of the estimates of $\beta_1$, $\beta_2$, $\gamma$ and $\sigma^2$ for varying grid density $m$ = 10, 20, 40, 80

The derivation of the maximum likelihood estimators has been developed in detail because it directly concerns the clear understanding and interpretation of the simulation study and its results. The calculation of the Fisher information matrix and its inverse is also interesting, among other issues, to study the asymptotic behavior of these estimators. This calculation, tedious and with a certain technical complexity in the Gompertz case, has not been considered here because it was not used explicitly in the simulation study (see Sect. 4), the main aim of this paper. This methodology has been discussed by several authors in different situations, such as concerning diffusion processes. For example, using continuous sampling, the Fisher information matrix was obtained in Gutiérrez et al. (2008b) for a I-CIR diffusion (the inverse Cox–Ingersoll–Ross diffusion model) to study the asymptotic behavior of the estimators. In the multivariate case, Dunn and Brisbin (1985) obtained the Fisher information matrix for the multivariate Ornstein–Uhlenbeck diffusion process based on discrete sampling.

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